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SPATIAL MOTION OF A ROD IN A VISCOUS FLUID FLOW

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Equations of spatial motion of a curved finite-length rod in a viscous fluid flow are derived. Analytical solutions of problems on the motion of a straight rod under conditions of pure shear, simple shear, and uniaxial extension of the fluid are obtained. Longitudinal stability of the straight rod during its spatial motion is considered. Effective viscosity of a suspension filled by rigid straight rods is evaluated.

Key words: *viscous fluid flow, straight rod, curved rod, elastic line.*

Differential equations of the elastic line for a thin elastic rod with different principal stiffnesses, which is in equilibrium under the action of forces applied to its end, was derived by G. R. Kirchhoff. This problem was considered in much detail in [1].

In the present paper, we consider the evolution of the shape of the elastic line of a rod under the action of a distributed load acting from the viscous fluid. The intensity of the distributed load depends on the spatial orientation of the rod. The main features of planar motion of a finite-length filament and rod in a viscous fluid flow were studied in [2, 3].

As one application of the results of solving the problem considered, we can mention processing of polymer materials filled by short reinforcing fibers of different nature (polyamide, glass, carbon, metal, etc.). Processing of these materials is accompanied by mixing of the polymer melt with a filler, destruction of the fibrous filler, orientation of the filler, formation of a near-wall layer, etc. The equations obtained can be used for analyzing the reptational motion of long-length biological objects in a continuous medium and also orientational effects of electromagnetorheological fluids.

1. Dynamic Equations. We consider an isolated spatial rod of an arbitrary shape in a laminar flow of a viscous fluid. In the case of bending of a thin rod, external forces acting on the side surface are small as compared to stresses arising inside the rod. The forces of inertia and gravity are negligibly small. The rod is wetted by the fluid, and no-slipping conditions are satisfied. On the side of the viscous fluid, the rod is affected by the friction force; the velocity field in the fluid remains unchanged thereby (effect of aeroelasticity is ignored). Elastic strains caused by extension or compression by the elastic axis of the rod are not taken into account. Both in the natural state and in the course of deformation, the rod has no sectors of considerable curvature, and the condition $\max(d/l, kd) \ll 1$ is fulfilled ($2l$ and d are the rod length and diameter; k is the curvature). The cross-sectional area is assumed to be small as compared to the total size of the rod and to remain unchanged during deformation, i.e., the pressure of longitudinal fibers on each other is absent. During bending, the cross sections remain planar and only turn by a certain angle with respect to the initial position.

We introduce the coordinate system (x, y, z), which is motionless in space (or "frozen" into the fluid). We denote the coordinates of the points of the elastic line of the rod s as x, y, z. The position of the curve s is described by the vector function $\boldsymbol{r}(s,t)$, $-l \leqslant s \leqslant l$ (t is the time). The directions x, y, z correspond to right-hand oriented trihedral $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$. We denote the vector of the tangent to the elastic line as $\boldsymbol{l}$ ($\boldsymbol{l} = \boldsymbol{r}_s$ and $|\boldsymbol{l}| = 1$), the normal vector as $\boldsymbol{n} = \boldsymbol{b} \times \boldsymbol{l}$, and the binormal vector as $\boldsymbol{b}$.

According to [4], the equations of equilibrium of the rod have the form

$$\boldsymbol{F}_s = -\boldsymbol{K}, \qquad \boldsymbol{M}_s + \boldsymbol{m} = \boldsymbol{F} \times \boldsymbol{l}, \tag{1.1}$$

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where $\boldsymbol{K} = A\boldsymbol{l}((\boldsymbol{V} - \boldsymbol{r}_t)\boldsymbol{l}) + B\boldsymbol{n}((\boldsymbol{V} - \boldsymbol{r}_t)\boldsymbol{n}) + B\boldsymbol{b}((\boldsymbol{V} - \boldsymbol{r}_t)\boldsymbol{b})$ is the linear density of external forces, $A = 2\pi\mu/\ln(0.952/\sqrt{c})$ is a coefficient characterizing the longitudinal component of the friction force, μ is the viscosity of the fluid, $\boldsymbol{m}$ is the moment of the external load per unit length, c is the volume concentration of the fibrous filler in the fluid, $B = 4\pi\mu/\ln(7.4/\mathrm{Re})$ is a coefficient characterizing the transverse component of the friction force, $\mathrm{Re} = \langle v\rangle\rho d/\mu$ is the Reynolds number, ρ is the density of the fluid, $\langle v\rangle$ is the characteristic velocity, $\boldsymbol{V}$ is the velocity of the fluid, and $\boldsymbol{M}$ is the moment; the subscripts indicate the corresponding derivatives.

In addition to the axial and transverse motion of the rod in the fluid [2], we take into account the rotation of its surface around the elastic axis with a circumferential velocity $v_\theta = 0.5d\chi_s\boldsymbol{r}_t\boldsymbol{l}$. The rod of diameter d rotates in a quasi-tube (whose walls are formed by the neighboring fibers) of radius $\langle r\rangle = d/(2.1\sqrt{c})$ filled by a viscous fluid [2]. The rod surface is affected by the shear stress $\tau_{\tau\theta} = 12\mu v_\theta\langle r\rangle/(d^2 - 4\langle r\rangle^2)$. Hence, the moment of the external load is determined by the expression $\boldsymbol{m} = 0.5\pi d^2\tau_{\tau\theta}\boldsymbol{l}$, where $\tau_{\tau\theta} = -25.2\mu\sqrt{c}v_\theta/[d(4 - 4.41c)]$ and $v_\theta = 0.5d\chi(\alpha x_t + \beta y_t + \gamma z_t)$. The transverse components of the moment of the external load are quantities of the highest orders of smallness; therefore, we assume that $\boldsymbol{mn} = 0$ and $\boldsymbol{mb} = 0$.

With allowance for the relations $\boldsymbol{F} = (\boldsymbol{Fl})\boldsymbol{l} + (\boldsymbol{Fn})\boldsymbol{n} + (\boldsymbol{Fb})\boldsymbol{b} = N\boldsymbol{l} + Q\boldsymbol{n} + P\boldsymbol{b}$, $\boldsymbol{l}_s = k\boldsymbol{n}$, $\boldsymbol{M} = EJ(\boldsymbol{l} \times \boldsymbol{l}_s - \boldsymbol{l}_0 \times \boldsymbol{l}_{0s}) + GJ_p(\chi - \chi_0)\boldsymbol{l}$ [3], $\boldsymbol{n}_s = -k\boldsymbol{l} + \chi\boldsymbol{b}$, $\boldsymbol{b}_s = -\chi\boldsymbol{n}$, $\boldsymbol{l} \times \boldsymbol{l} = 0$, $\boldsymbol{n} \times \boldsymbol{l} = -\boldsymbol{b}$, $\boldsymbol{l}_s \times \boldsymbol{l}_s = 0$, and $\boldsymbol{l} \times \boldsymbol{l}_{ss} = k_s\boldsymbol{b} - k\chi\boldsymbol{n}$, Eqs. (1.1) take the form

$$\boldsymbol{l}(N_s - kQ) + \boldsymbol{n}(Q_s + Nk + P\chi) + \boldsymbol{b}(P_s + \chi Q)$$
$$= -A\boldsymbol{l}((\boldsymbol{V} - \boldsymbol{r}_t)\boldsymbol{l}) - B\boldsymbol{n}((\boldsymbol{V} - \boldsymbol{r}_t)\boldsymbol{n}) - B\boldsymbol{b}((\boldsymbol{V} - \boldsymbol{r}_t)\boldsymbol{b}); \tag{1.2}$$

$$EJ[(k_0\chi_0 - k\chi)\boldsymbol{n} + (k_s - k_{0s})\boldsymbol{b}] + GJ_p[k(\chi - \chi_0)\boldsymbol{n} + (\chi_s - \chi_{0s})\boldsymbol{l}] + 0.5\pi d^2\tau_{\tau\theta}\boldsymbol{l} = -Q\boldsymbol{b} + P\boldsymbol{n}, \tag{1.3}$$

where χ is the torsion, N is the longitudinal force, Q and P are the components of the shear force, G is the shear modulus, $J_p = \pi d^4/32$ is the polar moment of inertia of the cross section, E is the modulus of elasticity, $J = \pi d^4/64$ is the moment of inertia of the cross section, $\boldsymbol{l}_0$ is the vector of the tangent line corresponding to the initial (natural) configuration of the rod, k_0 and χ_0 are the initial curvature and torsion, $\boldsymbol{V} - \boldsymbol{r}_t = (v_x - x_t)\boldsymbol{i} + (v_y - y_t)\boldsymbol{j} + (v_z - z_t)\boldsymbol{k}$, v_x, v_y, and v_z are the velocity components, $\boldsymbol{l} = \alpha\boldsymbol{i} + \beta\boldsymbol{j} + \gamma\boldsymbol{k}$, $\boldsymbol{n} = l\boldsymbol{i} + m\boldsymbol{j} + n\boldsymbol{k}$, $\boldsymbol{b} = \lambda\boldsymbol{i} + \mu\boldsymbol{j} + \nu\boldsymbol{k}$, and α, β, γ, l, m, n, λ, μ, and ν are the direction cosines of nine angles formed by the axes of the "moving" trihedral with the coordinate axes x, y, and z.

The cosines of the angles formed by the tangent line with the coordinate axes x, y, and z are $\alpha = x_s$, $\beta = y_s$, and $\gamma = z_s$, respectively.

Equations (1.2) and (1.3) should be supplemented by the orthogonality conditions

$$\begin{gathered}\alpha^2 + \beta^2 + \gamma^2 = 1, \qquad l^2 + m^2 + n^2 = 1, \qquad \lambda^2 + \mu^2 + \nu^2 = 1,\\ \alpha l + \beta m + \gamma n = 0, \qquad \alpha\lambda + \beta\mu + \nu\gamma = 0, \qquad l\lambda + m\mu + n\nu = 0.\end{gathered} \tag{1.4}$$

If there are no stresses in the rod at the initial time, the boundary conditions for the rod with free ends have the form

$$t = 0,\ \boldsymbol{r} = \boldsymbol{r}_0\colon\quad \boldsymbol{M} = \boldsymbol{F} = 0, \qquad t > 0,\ s = \pm l\colon\quad \boldsymbol{M} = \boldsymbol{F} = 0, \tag{1.5}$$

where $\boldsymbol{r}_0$ is the radius vector of the initial (natural) configuration of the rod.

Resolving Eq. (1.2) with respect to $\boldsymbol{V} - \boldsymbol{r}_t$ and differentiating both parts of the resultant expression with respect to s, we obtain the equation in a form more convenient for analysis (the functions of x, y, and z are eliminated):

$$\boldsymbol{V}_s - \boldsymbol{r}_{ts} = [-B^{-1}(Q_{ss} + N_sk + Nk_s + P_s\chi + P\chi_s) - A^{-1}k(N_s - kQ) + B^{-1}\chi(P_s + Q\chi)]\boldsymbol{n}$$
$$+ [kB^{-1}(Q_s + Nk + P\chi) - A^{-1}(N_{ss} - k_sQ - kQ_s)]\boldsymbol{l}$$
$$+ [-B^{-1}\chi(Q_s + Nk + P\chi) - B^{-1}(P_{ss} + \chi_sQ + \chi Q_s)]\boldsymbol{b}. \tag{1.6}$$

Here, $\boldsymbol{r}_{ts} = \boldsymbol{l}_t = \alpha_t\boldsymbol{i} + \beta_t\boldsymbol{j} + \gamma_t\boldsymbol{k} = (\alpha_tl + \beta_tm + \gamma_tn)\boldsymbol{n}$, $\boldsymbol{V}_s = (\boldsymbol{l}\nabla)\boldsymbol{V} = [((\boldsymbol{l}\nabla)\boldsymbol{V})\boldsymbol{n}]\boldsymbol{n} + [((\boldsymbol{l}\nabla)\boldsymbol{V})\boldsymbol{l}]\boldsymbol{l} + [((\boldsymbol{l}\nabla)\boldsymbol{V})\boldsymbol{b}]\boldsymbol{b}$, and $\nabla = \boldsymbol{i}\,\partial/\partial x + \boldsymbol{j}\,\partial/\partial y + \boldsymbol{k}\,\partial/\partial z$.

Thus, for 12 quantities (functions of s and t) P, Q, N, α, β, γ, l, m, n, λ, μ, and ν, we have 12 equations: Eqs. (1.3), (1.4), and (1.6).

According to (1.6), the change in orientation or shape of the rod is due to the velocity gradient, since the constant component of velocity ($v_x = \text{const}$, $v_y = \text{const}$, $v_z = \text{const}$) is caused by convective displacement of the rod along the corresponding coordinate axis without changing its configuration. Consequently, in studying conformal transformations, one can put the origin of the Cartesian coordinate system at an arbitrary point of the rod, e.g., in the middle of the elastic axis ($x = 0$, $y = 0$, $z = 0$, and $s = 0$).

If the flexural rigidity of the rod is low, so that the rod can be considered as a filament (see Sec. 5 and [3] for more detail), the equations of motion can be significantly simplified by assuming that $\boldsymbol{M} = 0$ and $\boldsymbol{F} = N\boldsymbol{l}$.

2. Motion of the Rod in a Simple Shear Flow. Let the x axis lie in the horizontal plane and correspond to the flow direction. The velocity components are $v_x = \gamma_- y$, $v_y = 0$, and $v_z = 0$ ($\gamma_- = \partial v_x/\partial y$ is the strain rate). We consider an arbitrarily oriented straight rod of length $2l$ whose middle coincides with the origin. Orientation is characterized by the functions $\alpha(t)$, $\beta(t)$, and $\gamma(t)$. The equalities $k = 0$ and $\chi = 0$ are valid, which yields $Q = 0$ and $P = 0$.

Using Eqs. (1.3)–(1.6), we obtain the following problem:

$$l\beta\gamma_- - \alpha_t l - \beta_t m - \gamma_t n = 0, \qquad \alpha\beta\gamma_- = -A^{-1}N_{ss}, \qquad \lambda\beta = 0,$$
$$t = 0\colon \quad \alpha = \alpha_0, \quad \beta = \beta_0, \quad \gamma = \gamma_0, \qquad t > 0, \ s = \pm l\colon \quad N = 0. \tag{2.1}$$

Here and in problems considered in Secs. 3 and 4, the orthogonality conditions (1.4) are not repeated.

It follows from the second equation in (2.1) that

$$N = 0.5A\alpha\beta\gamma_-(l^2 - s^2). \tag{2.2}$$

According to (2.2), there are no loads in the rod if the equality $\alpha\beta = 0$ is satisfied, which is the case when the rod is located either in the horizontal plane ($\beta = 0$) or in the vertical plane ($\alpha = 0$). The maximum force in the rod corresponds to the condition $\alpha^2 = \beta^2$, where the elastic axis lies in the plane passing through the z axis and is inclined to the x axis at an angle $\pm\pi/4$, which agrees with the results of [2].

In accordance with (2.2), we have $\beta \neq 0$ in the general case; therefore, we should assume that $\lambda = 0$ in the third equation in (2.1). In solving the problem, it is convenient to pass to the Euler angles θ, φ, and ψ. For the first three direction cosines, we have [5]

$$\alpha = \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi, \quad \beta = \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi, \quad \gamma = \sin\psi\sin\theta. \tag{2.3}$$

The first equation in (2.1) and the equation $\lambda = 0$ take the form

$$-\gamma_-\cos\theta\sin^2\psi + \psi_t = 0, \qquad \sin\varphi\sin\theta = 0. \tag{2.4}$$

To satisfy the second equation in (2.4), it suffices to put $\sin\varphi = 0$. In addition, we assume that $\cos\theta = C$, where C is a constant. Integrating the first equation in (2.4), with allowance for the condition $t = 0$, $\psi = \psi_0$, we obtain

$$\tan\,\psi = \tan\,\psi_0(1 + (\tan\,\psi_0)C\gamma_- t)^{-1}. \tag{2.5}$$

With allowance for the equalities $\sin\varphi = 0$, $\cos\varphi = 1$, and $\cos\theta = C$, using formulas (2.3), we obtain

$$\alpha = (1 + \tan^2\psi)^{-0.5}, \quad \beta = C\tan\,\psi(1 + \tan^2\psi)^{-0.5}, \quad \gamma = \tan\,\psi(1 - C^2)^{0.5}(1 + \tan^2\psi)^{-0.5}. \tag{2.6}$$

From the initial conditions (2.1), we find

$$\tan\,\psi_0 = \alpha_0^{-1}(\beta_0^2 + \gamma_0^2)^{0.5}, \qquad C = \beta_0(\beta_0^2 + \gamma_0^2)^{-0.5}. \tag{2.7}$$

The absolute value of the rod velocity is determined by the formula $v = |\boldsymbol{r}_t| = \sqrt{x_t^2 + y_t^2 + z_t^2}$. With allowance for the relations $x = \alpha s$, $y = \beta s$, and $z = \gamma s$, the formula for the absolute velocity of the rod end takes the form

$$V = \sqrt{\alpha_\tau^2 + \beta_\tau^2 + \gamma_\tau^2}, \tag{2.8}$$

where $V = v\Big|_{s=l}/(l\gamma_-)$ is the dimensionless velocity.

Figure 1 shows the velocity of the rod end versus the parameter α. The maximum of velocity is observed at the point $\alpha = 0$, which corresponds to the moment of intersection of the plane yz by the rod. The closer the rod to the plane xz, the lower its velocity. When approaching the point of static equilibrium ($\alpha = 1$), the velocity of the rod end vanishes.

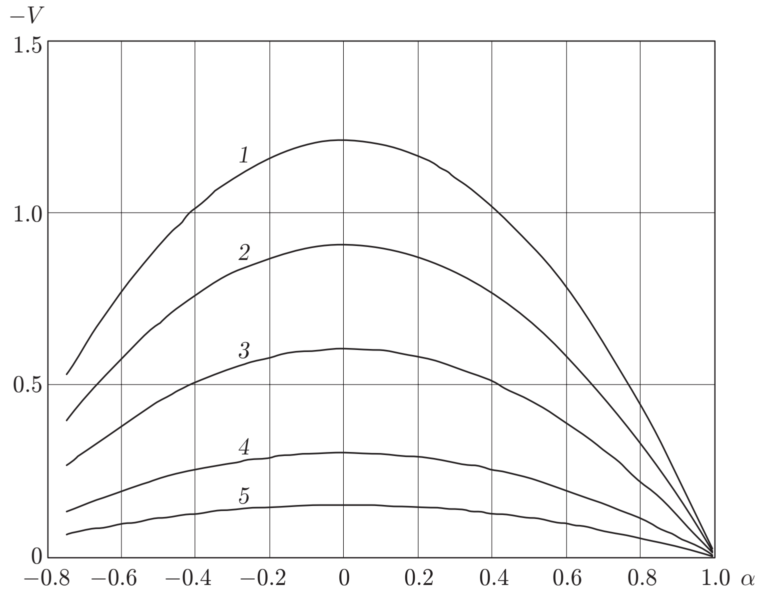

Fig. 1. Velocity of the rod end versus the parameter α for $\alpha_0 = -0.75$: $\beta_0 = 0.8$ (1), 0.6 (2), 0.4 (3), 0.2 (4), and 0.1 (5).

3. Motion of the Rod Under Conditions of Pure Shear of the Fluid. In pure shear, the velocity components are determined by the relations $v_x = \gamma_+ x$, $v_y = -\gamma_+ y$, and $v_z = 0$ ($\gamma_+ = \partial v_x/\partial x$ is the strain rate). With allowance for Eqs. (1.3)–(1.6) and assumptions of Sec. 2, the motion of the straight rod is described by the following equations:

$$l\alpha\gamma_+ - m\beta\gamma_+ - \alpha_t l - \beta_t m - \gamma_t n = 0, \qquad \alpha^2\gamma_+ - \beta^2\gamma_+ = -A^{-1}N_{ss}, \qquad \lambda\alpha - \mu\beta = 0,$$
$$t = 0\colon \quad \alpha = \alpha_0, \quad \beta = \beta_0, \quad \gamma = \gamma_0, \qquad t > 0, \ s = \pm l\colon \quad N = 0. \tag{3.1}$$

From the second equation in (3.1), we find the axial force

$$N = 0.5A\gamma_+(\alpha^2 - \beta^2)(l^2 - s^2). \tag{3.2}$$

It follows from expression (3.2) that the forces in the rod are compressive ($N < 0$) for $\beta^2 > \alpha^2$ and tensile ($N > 0$) for $\alpha^2 > \beta^2$. If the elastic axis of the rod lies in the plane passing through the z axis and is inclined to the x axis at an angle $\pi/4$, i.e., the equality $\beta^2 = \alpha^2$ is fulfilled, there is no force in the rod ($N = 0$). The maximum tensile and compressive forces in the rod are observed for $\beta = 0$ and $\alpha = 0$, respectively.

With allowance for relations (2.3), we obtain the problem

$$0.5\sin 2\psi \sin^{-1} 2\varphi + \cos\theta\varphi_\tau + \psi_\tau = 0, \qquad \cos\theta + \tan 2\varphi \cot \psi = 0,$$
$$\tau = 0, \qquad \varphi = \varphi_0, \qquad \psi = \psi_0, \tag{3.3}$$

where $\tau = \gamma_+ t$. For three unknown functions θ, φ, and ψ, we have only two equations; therefore, we fix the nutation angle $\cos\theta = C$, where C is a constant. The solution of problem (3.3) has the form

$$(1 - 0.5C^2)\ln|\tan\varphi/\tan\varphi_0| - 0.5(1 - C^2)(\cos 2\varphi - \cos 2\varphi_0) = -\tau. \tag{3.4}$$

Let us return to the functions α, β, and γ, using formulas (2.3), the second equation in (3.3), and the known trigonometric relations

$$\alpha = \frac{C\sqrt{1 + \tan^2\varphi}}{\sqrt{C^2(1 - \tan^2\varphi)^2 + 4\tan^2\varphi}},$$
$$\beta = \frac{C\tan\varphi\sqrt{1 + \tan^2\varphi}}{\sqrt{C^2(1 - \tan^2\varphi)^2 + 4\tan^2\varphi}}, \qquad \gamma = \frac{2\sqrt{1 - C^2}\tan\varphi}{\sqrt{C^2(1 - \tan^2\varphi)^2 + 4\tan^2\varphi}}. \tag{3.5}$$

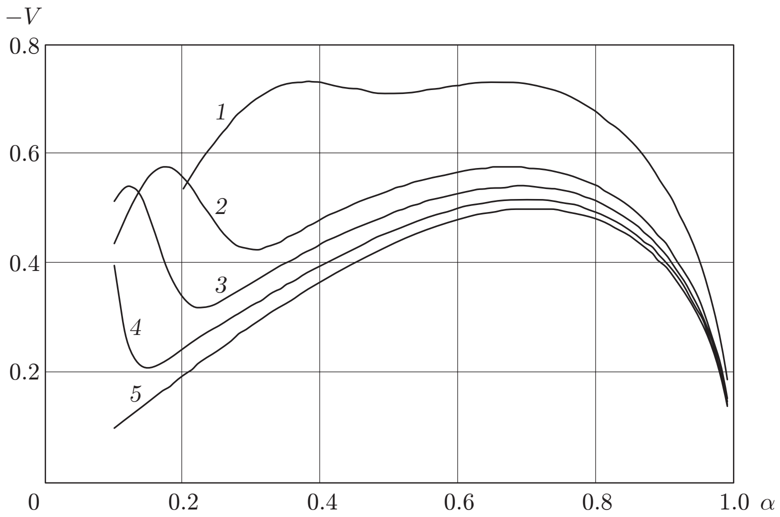

Fig. 2. Velocity of the rod end versus the parameter α for different initial positions of the rod: $\alpha_0 = 0.2$ and $\beta_0 = 0.9$ (1), $\alpha_0 = 0.1$ and $\beta_0 = 0.9$ (2), $\alpha_0 = 0.1$ and $\beta_0 = 0.8$ (3), $\alpha_0 = 0.1$ and $\beta_0 = 0.4$ (4), and $\alpha_0 = \beta_0 = 0.001$ (5).

From the initial conditions, we find the constants

$$\tan^2 \varphi_0 = (\beta_0/\alpha_0)^2, \qquad C^2 = 4\alpha_0^2\beta_0^2/[\alpha_0^2 + \beta_0^2 - (\alpha_0^2 - \beta_0^2)^2]. \tag{3.6}$$

In the two-dimensional case ($\gamma = 0$, $C = 1$, $\alpha_0^2 + \beta_0^2 = 1$), from (3.4)–(3.6), we obtain $\alpha^2 = \alpha_0^2/[\alpha_0^2 + \beta_0^2 \exp(-4\tau)]$, which agrees with the results of [3].

Figure 2 shows the velocity of the rod end [see (2.8)] versus the parameter α for different initial positions of the rod. The dependences in Fig. 2 differ significantly from similar dependences in the case of simple shear. An extremum is observed for $\alpha \approx 0.7$. The presence of a second extremum in the region $\alpha < 0.7$ depends on the initial position of the rod. Even if the rod is located close to the plane xz (curve 5), the velocity of its turning in the downstream direction is rather high; in this case, there is one extremum on the curve $V(\alpha)$.

4. Motion of the Rod Under Uniaxial Extension of the Fluid. In the case of rod motion under uniaxial extension of the fluid, the velocity components have the form [6] $v_x = \gamma_* x$, $v_y = -0.5\gamma_* y$, $v_z = -0.5\gamma_* z$ ($\gamma_* = \partial v_x/\partial x$ is the strain rate).

With allowance for the assumptions accepted in Sec. 2 for the straight rod, we obtain the following problem using Eqs. (1.3)–(1.6):

$$l\alpha\gamma_* - 0.5m\beta\gamma_* - 0.5n\gamma\gamma_* - \alpha_t l - \beta_t m - \gamma_t n = 0,$$

$$\alpha^2\gamma_* - 0.5\beta^2\gamma_* - 0.5\gamma^2\gamma_* = -A^{-1}N_{ss}, \qquad \lambda\alpha - 0.5\mu\beta - 0.5\nu\gamma = 0, \tag{4.1}$$

$$t = 0\colon \quad \alpha = \alpha_0, \quad \beta = \beta_0, \quad \gamma = \gamma_0, \qquad t > 0, \; s = \pm l\colon \quad N = 0.$$

The solution of the second equation in (4.1) has the form

$$N = 0.25A\gamma_*(3\alpha^2 - 1)(l^2 - s^2). \tag{4.2}$$

According to (4.2), there is no force in the rod ($N = 0$) if its elastic axis lies on the surface of a round cone with the apex in the origin and an apex angle whose cosine is $\alpha = 1/\sqrt{3}$. If the rod axis is located inside this cone ($\alpha > 1/\sqrt{3}$), the forces are tensile; if the rod axis is located outside the cone ($\alpha < 1/\sqrt{3}$), the forces are compressive.

We write the first and third equations in (4.1) using the angles ψ, θ, and φ [see (2.3)]:

$$1.5 \sin\psi\cos\psi + \cos\theta\varphi_\tau + \psi_\tau = 0, \qquad \sin\theta\sin\varphi = 0. \tag{4.3}$$

Here, $\tau = \gamma_* t$. In the second equation in (4.3), we assume that $\sin\varphi = 0$ and $\theta = \theta_0 = \text{const}$. The solution of the first equation with allowance for the initial condition $\tau = 0$, $\psi = \psi_0$ has the form

$$\tan\psi = \tan\psi_0 \exp(-1.5\tau). \tag{4.4}$$

Using relations (2.3), we come back to the variables α, β, and γ:

$$\alpha = \cos\psi, \qquad \beta = \cos\theta_0 \sin\psi, \qquad \gamma = \sin\theta_0 \sin\psi. \tag{4.5}$$

The constants θ_0 and ψ_0 are found from conditions (4.1): $\tan\theta_0 = \gamma_0/\beta_0$ and $\cos\psi_0 = \alpha_0$.

According to (4.4) and (4.5), in the case of long-time deformation, the elastic axis of the rod coincides with the x axis ($\alpha = 1$, $\beta = \gamma = 0$ as $\tau \to \infty$) regardless of the initial orientation. In the course of evolution, the elastic axis moves in the plane passing through the x axis and the straight line corresponding to the initial position of the rod.

The absolute value of velocity of the rod end (2.8) is described by the expression $V = -1.5\alpha\sqrt{1-\alpha^2}$, is independent of the functions β and γ, and has one extremum $V = -0.75$ at $\alpha = 1/\sqrt{2}$.

5. Longitudinal Stability of the Rod. According to the results of Secs. 2–4, the distribution of the axial force along the rod is described by a parabola, and the absolute value of the force and its sign depend on rod orientation; therefore, in the general case, we can write

$$N = -D(\alpha, \beta)(l^2 - s^2), \tag{5.1}$$

where D is a function depending on orientation and type of the flow [see (2.2), (3.2), and (4.2)].

In the case of compressive forces ($D > 0$), buckling is possible due to longitudinal bending of the rod, which, in the case of high-polymer fibers (glass, carbon), leads to their destruction. In [2], stability was studied by introducing small perturbations of the shape and by using linearized equations. We study the planar (flexural) form of buckling [7].

The distributed load of the axial friction force is symmetric; therefore, we restrict ourselves to the sector $0 \leqslant s \leqslant l$, sealing half of the rod. As the rod axis is deflected by an angle ξ, the shear force Q with allowance for (5.1) is determined by the integral

$$Q = -\sin\xi \int_s^l \frac{\partial N}{\partial s}\, ds = -0.5D(l^2 - s^2)\xi.$$

Here $\sin\xi \approx \xi$ for $|\xi| \ll 1$; the quantity $\partial N/\partial s$ characterizes the intensity of the axial distributed load.

The bending moment is $M = EJ\xi_s$. Hence, the shear force is $Q = M_s = EJ\xi_{ss}$. From here, we obtain the differential equation for the deflection:

$$\xi_{ss} + D(l^2 - s^2)\xi/(2EJ) = 0. \tag{5.2}$$

The boundary conditions (zero moment at the free end and absence of deflection at the sealed end) are written as

$$s = l, \quad \xi_s = 0; \qquad s = 0, \quad \xi = 0. \tag{5.3}$$

The solution of problem (5.2), (5.3) is not expressed via elementary functions; therefore, the eigenvalues $\lambda = Dl^4/(2EJ)$ were found numerically: $\lambda_1 = 5.122$, $\lambda_2 = 39.66$, $\lambda_3 = 106.249$, $\lambda_4 = 204.86, \ldots$. Let us denote $D_1 = 2\lambda_1 EJ/l^4$. Hence, the rod loses stability for $D > D_1$ and retains stability for $D < D_1$. If the condition $D < D_1$ is satisfied for $D = D_{\max}$ (the value $D_{\max}$ corresponds to the maximum compressive force), the rod retains stability for an arbitrary spatial orientation. For instance, in the case of a uniaxial flow, according to (4.2), we have $D_{\max} = D\Big|_{\alpha=0} = 0.25A\gamma_*$.

Twisting of low-modulus fibers into a ball during suspension mixing was experimentally observed [8], which can be attributed to instability of the neutral equilibrium position in the rod under conditions of a simple shear flow [2, 3]. Performing rotating motion, the rod passes through positions corresponding to $\alpha > 0$ and $\beta < 0$, where the axial compressive force acts ($N < 0$). Viscous friction forces prevent buckling of the middle of the rod; therefore, there are all conditions for obtaining higher forms of longitudinal bending, corresponding to $\lambda_2, \lambda_3, \ldots$. Indeed, the numerical analysis of the problems [2, 3] demonstrated the possibility of obtaining a saw-tooth shape of the rod under significant compressive forces. Mixing of polyurethane fibers in a polyacrylate matrix was performed under the following conditions [8]: $d = 30$ μm, $c = 0.1$, $\mu = 0.1$ Pa$\cdot$sec, $E = 0.5$ MPa, $\gamma = 200$ sec^{-1}, and $2l = 1$ mm. In this case, we have $D_{\max} = 0.25A\gamma_- = 9.46$ Pa and $D_1 = 3.31$ Pa; hence, the rod (polyurethane fibers) loses stability ($D_{\max} > D_1$).

6. Viscosity of a Suspension Filled by Rigid Rods. The flow of a system filled by flexible or rigid rods requires additional energy for overcoming viscous friction forces in the flow around these rods. Consequently, the effective viscosity of the system is greater than the viscosity of the pure fluid.

For a rod of length $2l$, the energy of viscous friction W is determined by the integral [3]

$$W = \int_{-l}^{l} (\boldsymbol{V} - \boldsymbol{r}_t)\boldsymbol{K}\, ds.$$

Taking into account the first equation in (1.1) and resolving (1.2) with respect to the velocity difference $\boldsymbol{V} - \boldsymbol{r}_t = -B^{-1}(Q_s + Nk + P\chi)\boldsymbol{n} - A^{-1}(N_s - kQ)\boldsymbol{l} - B^{-1}(P_s + \chi Q)\boldsymbol{b}$, we obtain the following equation for a curved rod:

$$W = \int_{-l}^{l} [B^{-1}(Q_s + Nk + P\chi)^2 + A^{-1}(N_s - kQ)^2 + B^{-1}(P_s + \chi Q)^2]\, ds.$$

In the case of a rather rigid straight rod ($k = 0$, $\chi = 0$, $P = 0$, $Q = 0$), we have

$$W = A^{-1} \int_{-l}^{l} N_s^2\, ds.$$

The total energy losses W_Σ caused by the flow around all rods are determined by the relation [3]

$$W_\Sigma = \frac{2V_+ c}{\pi d^2 l A} \int_{-l}^{l} N_s^2\, ds, \tag{6.1}$$

where V_+ is the volume of the suspension.

In the case of simple shear, Eqs. (2.2), (2.5)–(2.7), and (6.1) yield

$$W_\Sigma = 4V_+ c A l^2 \gamma_-^2 \alpha^2 \beta^2/(3\pi d^2), \tag{6.2}$$

where $\tau = \gamma_- t$, $\beta = \beta_0/\sqrt{(\alpha_0 + \beta_0\tau)^2 + 1 - \alpha_0^2}$, and $\alpha = \beta(\alpha_0 + \beta_0\tau)/\beta_0$.

The expression for effective viscosity under conditions of simple shear with allowance for additional energy losses for the flow around the rods is

$$\mu_+ = (\tau_{xy}\gamma_- V_+ + W_\Sigma)/(\gamma_-^2 V_+), \tag{6.3}$$

where $\tau_{xy} = \mu\gamma_-$ is the shear stress.

For a suspension filled by rods of identical length, diameter, and orientation, Eqs. (6.2) and (6.3) yield

$$\mu_+ = \mu\Big(1 + \frac{8}{3}\, \frac{l^2 c}{d^2 \ln(0.952/\sqrt{c}\,)}\, \alpha^2\beta^2\Big). \tag{6.4}$$

According to (6.4), if the rods are parallel to the plane xz ($\beta = 0$), the system viscosity is minimum. The system viscosity is maximum for $\alpha = \pm 1/\sqrt{2}$, $\beta = \pm 1/\sqrt{2}$, and $\gamma = 0$.

In the case of a polydisperse filler, viscosity is determined by the expression

$$\mu_+ = \mu\Big(1 + \frac{8c}{3\ln(0.952/\sqrt{c}\,)} \sum_{i=1}^{m} \frac{l_i^2}{d_i^2}\, \psi_i \alpha_i^2 \beta_i^2\Big), \tag{6.5}$$

where $i = 1, \ldots, m$ is the number of the fraction and ψ_i is the relative number of rods of the ith fraction $\Big(\sum_{i=1}^{m} \psi_i = 1\Big)$.

In simple shear, the position of neutral equilibrium is unstable; therefore, the rods rotate with a nonuniform velocity in the case of long-time deformation [2, 3]. Orientation acquires an isotropic character. We find the system viscosity for a random orientation. Because of the evenness of function (6.2), $W_\Sigma(\alpha) = W_\Sigma(-\alpha)$ and $W_\Sigma(\beta) = W_\Sigma(-\beta)$, we confine ourselves to the first octant $\theta \in [0, \pi/2]$, $\varphi \in [0, \pi/2]$ (θ and φ are spherical coordinates). In the directions θ_i and φ_i, we identify a two-dimensional sector with angles $\Delta\theta = \pi/(2m)$ and $\Delta\varphi = \pi/(2m)$ and consider l and d as functions of the angles θ_i and φ_i. For equiprobable orientation of the rods, the function ψ in (6.5) is determined as the ratio of the volumes of the selected cone and 1/8 of the sphere of unit radius:

$$\psi = \Delta\theta\Delta\varphi \sin\theta \Big/ \int_0^{\pi/2} d\varphi \int_0^{\pi/2} \sin\theta\, d\theta = \frac{2}{\pi}\, \Delta\theta\Delta\varphi \sin\theta.$$

In this case, we have

$$\lim_{\substack{\Delta\theta\to 0\\ \Delta\varphi\to 0,\ m\to\infty}} \sum_{i=1}^{m} \Big(\frac{l(\varphi_i,\theta_i)}{d(\varphi_i,\theta_i)}\Big)^2 \frac{2\Delta\theta\Delta\varphi\sin\theta_i}{\pi}\,\alpha^2\beta^2 = \frac{2}{\pi}\int_0^{\pi/2}\int_0^{\pi/2}\Big(\frac{l}{d}\Big)^2\alpha^2\beta^2\sin\theta\,d\theta\,d\varphi.$$

For a monodisperse filler (l = const and d = const), the integral, with allowance for the ratio $\alpha^2\beta^2 = (\cos\theta)^2 \times (\sin\theta\sin\varphi)^2$, is $l^2/(15d^2)$, and expression (6.5) takes the form

$$\mu_+ = \mu\Big(1 + \frac{8}{45}\,\frac{l^2}{d^2}\,\frac{c}{\ln(0.952/\sqrt{c}\,)}\Big).$$

In the case of pure shear, with allowance for expressions (3.2) and (6.1) and results of [3], we obtain the relations

$$W_\Sigma = \frac{4V_+cl^2}{3\pi d^2}\,A\gamma_+^2(\alpha^2-\beta^2)^2, \qquad \mu_+ = \frac{\sigma_{xx}\gamma_+V_+ + W_\Sigma}{4\gamma_+^2V_+},$$

where $\sigma_{xx} = 4\mu\gamma_+$; the functions $\alpha(t)$ and $\beta(t)$ are determined in (3.4)–(3.6).

For a monodisperse system, effective viscosity is determined as

$$\mu_+ = \mu\Big(1 + \frac{2}{3}\,\frac{l^2c}{d^2\ln(0.952/\sqrt{c}\,)}\,(\alpha^2-\beta^2)^2\Big). \tag{6.6}$$

In the case of a polydisperse filler, we have

$$\mu_+ = \mu\Big(1 + \frac{2c}{3\ln(0.952/\sqrt{c}\,)}\sum_{i=1}^{m}\frac{l_i^2}{d_i^2}\,\psi_i(\alpha_i^2-\beta_i^2)^2\Big). \tag{6.7}$$

Under uniaxial extension, with allowance for expressions (4.2), (4.4), (4.5), and (6.1) and results of [3, 9], we obtain the relations

$$W_\Sigma = \frac{4V_+cl^2}{3\pi d^2}\,A\gamma_*^2(3\alpha^2-1)^2, \qquad \mu_+ = \frac{\sigma_{xx}\gamma_*V_+ + W_\Sigma}{3\gamma_*^2V_+}, \tag{6.8}$$

where $\alpha^2 = \alpha_0^2/[\alpha_0^2 + (1-\alpha_0^2)\exp(-3\tau)]$, $\tau = \gamma_*t$, and $\sigma_{xx} = 3\mu\gamma_*$ is the tensile stress.

From relations (6.8), we obtain the following expression for a monodisperse system:

$$\mu_+ = \mu\Big(1 + \frac{2}{9}\,\frac{l^2c}{d^2\ln(0.952/\sqrt{c}\,)}\,(3\alpha^2-1)^2\Big). \tag{6.9}$$

According to (6.9), the system viscosity is minimum for $\alpha^2 = 1/3$ and maximum for $\alpha^2 = 1$.

In the case of a polydisperse system, we have

$$\mu_+ = \mu\Big(1 + \frac{2c}{9\ln(0.952/\sqrt{c}\,)}\sum_{i=1}^{m}\frac{l_i^2}{d_i^2}\,\psi_i(3\alpha_i^2-1)^2\Big). \tag{6.10}$$

Here, $\alpha_i^2 = \alpha_{i0}^2/[\alpha_{i0}^2 + (1-\alpha_{i0}^2)\exp(-3\tau)]$.

Independent of the initial orientation, formulas (6.6), (6.7), (6.9), and (6.10) imply an asymptotic increase in viscosity with time, since $\alpha = 1$ and $\beta = 0$ as $\tau\to\infty$. The position of static equilibrium of the rods in the flows under extension is stable.

A comparison of the results obtained with the results for planar motion [3] shows that, other conditions being identical, the allowance for the spatial positions of the rods ($\gamma \neq 0$) leads to a decrease in effective viscosity of the suspension.

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